

FOURIER TRANSFORMS ON AN AMALGAM TYPE SPACE

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ABSTRACT. We introduce an amalgam type space, a subspace of $L^1(\mathbb{R}_+)$. Integrability results for the Fourier transform of a function with the derivative from such an amalgam space are proved. As an application we obtain estimates for the integrability of trigonometric series.

1. INTRODUCTION

Let us start with a brief overview of the (100 years) old problem of integrability of trigonometric series. Given a trigonometric series

$$(1) \quad a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

find assumptions on the sequences of coefficients $\{a_n\}, \{b_n\}$ under which the series is the Fourier series of an integrable function. Frequently, the series

$$(2) \quad a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx$$

and

$$(3) \quad \sum_{n=1}^{\infty} b_n \sin nx$$

are investigated separately, since there is a difference in their behavior. Usually, integrability of (3) requires additional assumptions. However, one of the basic assumptions is that the sequence $\{a_n\}$ or $\{b_n\}$ is of bounded variation, written $\{a_n\} \in bv$ or $\{b_n\} \in bv$, that is, satisfies the condition (for $\{a_n\}$; similarly for $\{b_n\}$)

$$(4) \quad \sum_{n=1}^{\infty} |\Delta a_n| < \infty,$$

where $\Delta a_n = a_n - a_{n+1}$ and similarly for Δb_n .

One of the strongest known conditions that ensures (along with certain natural assumptions) the integrability of trigonometric series, pulled in [1, 5], can be described as follows. Let the space of sequences $\{d_n\}$ be endowed with the norm

$$(5) \quad \|\{d_n\}\|_{a_{1,2}} = \sum_{m=0}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\sum_{n=j2^m}^{(j+1)2^m-1} |d_n| \right]^2 \right\}^{1/2} < \infty.$$

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It is of amalgam nature; the reader can consult on the theory of various amalgam spaces in [6, 9, 10].

It is proved in [1] and [5] that if the coefficients $\{a_n\}$ in (2) tend to 0 as $n \rightarrow \infty$ and the sequence $\{\Delta a_n\}$ is in $a_{1,2}$, then (2) represents an integrable function on $[0, \pi]$. In parallel, if $\{\Delta b_n\} \in a_{1,2}$, then (3) represents an integrable function on $[0, \pi]$ if and only if

$$(6) \quad \sum_{n=1}^{\infty} \frac{|b_n|}{n} < \infty.$$

Let us introduce the function space $A_{1,2}$ similar to (5). We say that a locally integrable function g defined on \mathbb{R}_+ belongs to $A_{1,2}$ if

$$(7) \quad \|g\|_{A_{1,2}} = \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\int_{j2^m}^{(j+1)2^m} |g(t)| dt \right]^2 \right\}^{1/2} dx < \infty.$$

This space is of amalgam nature as well, since each of the values we integrate after denotes the norm in the Wiener amalgam space $W(L^1, \ell^2)$ for functions $2^m g(2^m t)$, where ℓ^p is a space of sequences $\{d_j\}$ endowed with the norm

$$\|\{d_j\}\|_{\ell^2} = \left(\sum_{j=1}^{\infty} |d_j|^2 \right)^{1/2}$$

and the norm of a function $g : \mathbb{R}_+ \rightarrow \mathbb{C}$ from the amalgam space $W(L^1, \ell^2)$ is taken as

$$\left\| \left\{ \int_j^{j+1} |g(t)| dt \right\} \right\|_{\ell^2}.$$

In other words, we can rewrite (7) as follows:

$$(8) \quad \|g\|_{A_{1,2}} = \sum_{m=-\infty}^{\infty} \|2^m g(2^m \cdot)\|_{W(L^1, \ell^2)} < \infty.$$

The paper is organized as follows. Section 2 deals with a preliminary results on embedding of $A_{1,2}$ in L^1 . In Section 3 we prove our main results on integrability of the Fourier transforms. Then, in Section 4, we use some of the obtained results to get an extension of the above mentioned results on integrability of trigonometric series ([1, 5]).

Of course, the authors are aware of the multidimensional generalization in [2]. A multidimensional version of our results is in work and will appear elsewhere.

Here and in what follows $\varphi \lesssim \psi$ means that $\varphi \leq C\psi$ with C being an absolute constant.

2. $A_{1,2}$ IS A SUBSPACE OF $L^1(\mathbb{R}_+)$

We start with proving that the considered space is a subspace of $L^1(\mathbb{R}_+)$.

Lemma 1. *There holds $A_{1,2} \subset L^1(\mathbb{R}_+)$.*

Proof. We have

$$\begin{aligned} \ln \frac{3}{2} \int_0^\infty |g(t)| dt &= \int_0^\infty \frac{1}{x} \int_{2/x}^{3/x} |g(t)| dt dx \\ &\leq \sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \int_{2/x}^{3/x} |g(t)| dt dx. \end{aligned}$$

Since $\frac{2}{x} \geq 2^{-m}$ and $\frac{3}{x} \leq 3(2^{-m})$, the right-hand side does not exceed

$$\sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \int_{2^{-m}}^{3(2^{-m})} |g(t)| dt dx,$$

which, in turn, does not exceed

$$3 \sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \sum_{j=1}^\infty \frac{1}{j} \int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)| dt dx.$$

Indeed, the integral over $(2^{-m}, 3(2^{-m}))$ can be split into two ones over $(2^{-m}, 2(2^{-m}))$ and $(2(2^{-m}), 3(2^{-m}))$, and the first one is the first summand of the sum in j , while the second one is twice the second summand of that sum.

Applying now the Schwarz-Cauchy-Bunyakovskii inequality yields

$$\int_0^\infty |g(t)| dt \lesssim \sum_{m=-\infty}^\infty \left\{ \sum_{j=1}^\infty \left[\int_{j2^m}^{(j+1)2^m} |g(t)| dt \right]^2 \right\}^{1/2} dx,$$

the desired bound. \square

Remark 1. The following different form of Lemma 1 is useful in applications: if $f' \in A_{1,2}$, then f is of bounded variation, that is, $f' \in L^1(\mathbb{R}_+)$.

Before proceeding to estimates of Fourier transforms let us compare $A_{1,2}$ with a space important in such problems. In the study of integrability properties of the Fourier transform, the following T -transform of a function g defined on $(0, \infty)$ is of importance

$$(9) \quad Tg(t) = \int_0^{t/2} \frac{g(t-s) - g(t+s)}{s} ds = \int_{t/2}^{3t/2} \frac{g(s)}{t-s} ds,$$

where the integral is understood in the Cauchy principal value sense.

Note that the T -transform is related to the Hilbert transform given by

$$(10) \quad Hg(t) = \int_0^\infty \frac{g(t-s) - g(t+s)}{s} ds = \int_0^\infty \frac{g(s)}{t-s} ds.$$

This is revealed and discussed in [11], [8], etc. For example, it was shown in the mentioned works that for the Hilbert transform Hg of an integrable odd function, one has for $x > 0$

$$Hg(x) = Tg(x) + O(\Gamma(x)),$$

where $\int_{\mathbb{R}_+} |\Gamma(x)| dx \leq \int_{\mathbb{R}_+} |g(x)| dx$. On the other hand, let us remark that there is an essential difference between H and T -transforms since, e.g., for the characteristic function $\chi_{[0,d]}$, $d > 0$, we have ([8])

$$\|H\chi[0, \delta]\|_{L^1(\mathbb{R})} = \infty \quad \text{and} \quad \|T\chi[0, \delta]\|_{L^1(\mathbb{R}_+)} = \delta \ln 3.$$

Now, the space BT that has proved to be of importance and is related to the real Hardy space, is the space of all the functions that are integrable along with its T -transform. Indeed (see [11]), a function of bounded variation on \mathbb{R}_+ with the derivative from BT has the integrable cosine Fourier transform. As for the sine transform, an analog of (6) should be added.

Note that the classes BT and $A_{1,2}$ are incomparable. Counterexamples for sequences can be found in [1] and [7]).

3. MAIN RESULTS

We study, for $\gamma = 0$ or 1 , the Fourier transform

$$\widehat{f}_\gamma(x) = \int_0^\infty f(t) \cos\left(xt - \frac{\pi\gamma}{2}\right) dt.$$

It is clear that \widehat{f}_γ represents the cosine Fourier transform in the case $\gamma = 0$ while taking $\gamma = 1$ gives the sine Fourier transform.

Theorem 2. *Let f be locally absolutely continuous on \mathbb{R}_+ and vanishing at infinity, that is, $\lim_{t \rightarrow \infty} f(t) = 0$, and $f' \in A_{1,2}$. Then for $x > 0$*

$$(11) \quad \widehat{f}_\gamma(x) = \frac{1}{x} f\left(\frac{\pi}{2x}\right) \sin \frac{\pi\gamma}{2} + \theta\Gamma(x),$$

where $\gamma = 0$ or 1 , and $\|\Gamma\|_{L^1(\mathbb{R}_+)} \lesssim \|f'\|_{A_{1,2}}$.

Proof. Splitting the integral and integrating by parts, we obtain

$$\widehat{f}_\gamma(x) = -\frac{1}{x} f\left(\frac{\pi}{2x}\right) \sin \frac{\pi}{2}(1-\gamma) + \int_0^{\frac{\pi}{2x}} f(t) \cos\left(xt - \frac{\pi\gamma}{2}\right) dt - \frac{1}{x} \int_{\frac{\pi}{2x}}^\infty f'(t) \sin\left(xt - \frac{\pi\gamma}{2}\right) dt.$$

Further,

$$\begin{aligned}
& \int_0^{\frac{\pi}{2x}} f(t) \cos\left(xt - \frac{\pi\gamma}{2}\right) dt \\
&= \int_0^{\frac{\pi}{2x}} [f(t) - f(\frac{\pi}{2x})] \cos\left(xt - \frac{\pi\gamma}{2}\right) dt + \int_0^{\frac{\pi}{2x}} f(\frac{\pi}{2x}) \cos\left(xt - \frac{\pi\gamma}{2}\right) dt \\
&= - \int_0^{\frac{\pi}{2x}} \left[\int_t^{\frac{\pi}{2x}} f'(s) ds \right] \cos\left(xt - \frac{\pi\gamma}{2}\right) dt + \frac{1}{x} f(\frac{\pi}{2x}) \sin \frac{\pi}{2} (1 - \gamma) + \frac{1}{x} f(\frac{\pi}{2x}) \sin \frac{\pi\gamma}{2} \\
&= \frac{1}{x} f(\frac{\pi}{2x}) \sin \frac{\pi\gamma}{2} + \frac{1}{x} f(\frac{\pi}{2x}) \sin \frac{\pi}{2} (1 - \gamma) + O \left(\int_0^{\frac{\pi}{2x}} s |f'(s)| ds \right).
\end{aligned}$$

Since

$$(12) \quad \int_0^\infty \int_0^{\frac{\pi}{2x}} s |f'(s)| ds dx = \frac{\pi}{2} \int_0^\infty |f'(s)| ds,$$

it follows from Lemma 1 that to prove the theorem it remains to estimate

$$(13) \quad \int_0^\infty \frac{1}{x} \left| \int_{\frac{\pi}{2x}}^\infty f'(t) \sin\left(xt - \frac{\pi\gamma}{2}\right) dt \right| dx.$$

We can study

$$(14) \quad \sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^\infty f'(t) \sin\left(xt - \frac{\pi\gamma}{2}\right) dt \right| dx$$

instead. Indeed,

$$\begin{aligned}
(15) \quad \sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^{\frac{\pi}{2x}} f'(t) \sin\left(xt - \frac{\pi\gamma}{2}\right) dt \right| dx &\leq \sum_{m=-\infty}^\infty \int_{2^m}^{2^{m+1}} \frac{1}{x} \int_{\frac{1}{x}}^{\frac{\pi}{2x}} |f'(t)| dt dx \\
&\leq \int_0^\infty |f'(t)| dt.
\end{aligned}$$

Here and in other estimates with (14) there is no difference between the sine and cosine, therefore all will follow from the next result the statement and the proof of which are inspired by Lemma 2 in [1].

Lemma 2. *Let g be an integrable function on \mathbb{R}_+ . Then for $m = 0, \pm 1, \pm 2, \dots$*

$$(16) \quad \int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt \right| dx \lesssim \left(\sum_{j=1}^{\infty} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)| dt \right]^2 \right)^{1/2}.$$

Proof of Lemma 2. We have

$$\int_{2^m}^{2^{m+1}} \frac{1}{x} \left| \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt \right| dx \lesssim \int_{2^m}^{2^{m+1}} \left| \frac{\sin 2^{-m}x}{x} \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt \right| dx.$$

By the Schwarz-Cauchy-Bunyakovskii inequality the right-hand side does not exceed

$$(17) \quad \begin{aligned} & \left(\int_{2^m}^{2^{m+1}} dx \right)^{1/2} \left(\int_{2^m}^{2^{m+1}} \left| \frac{\sin 2^{-m}x}{x} \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt \right|^2 dx \right)^{1/2} \\ &= 2^{m/2} \left(\int_{2^m}^{2^{m+1}} \left| \frac{\sin 2^{-m}x}{x} \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt \right|^2 dx \right)^{1/2}. \end{aligned}$$

Denoting

$$S_m(x) = \frac{\sin 2^{-m}x}{x}; \quad G_m(x) = \int_{2^{-m}}^{\infty} g(t) e^{-ixt} dt,$$

we have to estimate

$$2^{m/2} \left(\int_{2^m}^{2^{m+1}} |S_m(x) G_m(x)|^2 dx \right)^{1/2}.$$

The product $S_m(x) G_m(x)$ is the convolution of the inverse Fourier transforms, and we get

$$(18) \quad 2^{m/2} \left(\int_{2^m}^{2^{m+1}} |\check{S}_m * \check{G}_m(x)|^2 dx \right)^{1/2}.$$

Let us take into account that

$$(19) \quad \int_0^{\infty} \frac{\sin ax}{x} \cos yx dx = \begin{cases} \frac{\pi}{2}, & y < a \\ \frac{\pi}{4}, & y = a \\ 0, & y > a \end{cases}$$

Remark 3. This formula goes back to Fourier, see, e.g., Remark 12 in the cited literature of [4].

Since all our estimates will be for integrals, the value at one point is of no importance. Therefore, using (19) for

$$\check{G}_m(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin 2^{-m}u}{u} \cos xu \, du,$$

we may consider $\check{S}_m(x)$ to be 1 for $x < 2^{-m}$ and zero otherwise.

Further,

$$G_m(x) = \sum_{j=1}^{\infty} \int_{j2^{-m}}^{(j+1)2^{-m}} g(t)e^{-ixt} \, dt = \sum_{j=1}^{\infty} G_{m,j}(x),$$

where

$$G_{m,j}(x) = \int_{j2^{-m}}^{(j+1)2^{-m}} g(t)e^{-ixt} \, dt.$$

Correspondingly,

$$\check{G}_m(x) = \sum_{j=1}^{\infty} g_{m,j}(x),$$

with $g_{m,j}(x) = g(x)$ when $j2^{-m} \leq x \leq (j+1)2^{-m}$ and zero otherwise.

Recall now that Young's inequality for convolution reads as follows (see, e.g., [12, Ch.V, §1]): If $\varphi \in L^r(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$, then for $\frac{1}{r} + \frac{1}{q} = \frac{1}{p} + 1$, $1 \leq p, q, r \leq \infty$,

$$(20) \quad \|\varphi * \psi\|_p \leq \|\varphi\|_r \|\psi\|_q.$$

To apply these, we may regard the supports I_j of $\check{S}_m(x) * \check{G}_{m,j}$ as non-overlapping. Otherwise, we can consider separately sums over j -s odd only and over j -s even, as in the proof of Lemma 2 in [1], arrive at the same upper bounds and then combine them.

Thus,

$$\begin{aligned} 2^{m/2} \left(\int_{2^m}^{2^{m+1}} |\check{S}_m * \check{G}_m(x)|^2 \, dx \right)^{1/2} &\leq 2^{m/2} \left(\int_{\mathbb{R}} |\check{S}_m * \check{G}_m(x)|^2 \, dx \right)^{1/2} \\ &\leq 2^{m/2} 2 \left(\sum_{j=1}^{\infty} \int_{I_j} |\check{S}_m * \check{G}_m(x)|^2 \, dx \right)^{1/2} \lesssim 2^{m/2} \left(\sum_{j=1}^{\infty} \int_{\mathbb{R}} |\check{S}_m * \check{G}_m(x)|^2 \, dx \right)^{1/2}. \end{aligned}$$

Applying Young's inequality with $\varphi = \check{S}_m$ and $\psi = g_{m,j}$, $q = 1$ and $p = r = 2$, we obtain

$$2^{m/2} \left(\sum_{j=1}^{\infty} \int_{\mathbb{R}} |\check{S}_m * \check{G}_m(x)|^2 dx \right)^{1/2} \lesssim 2^{m/2} \left(\sum_{j=1}^{\infty} \|\check{S}_m\|_2^2 \|\check{g}_{m,j}\|_1^2 \right)^{1/2}.$$

Since

$$\|\check{S}_m\|_2 = \left(\int_0^{2^{-m}} dx \right)^{1/2} = 2^{-m/2},$$

we get the required bound

$$\left(\sum_{j=1}^{\infty} \left[\int_{j2^{-m}}^{(j+1)2^{-m}} |g(t)| dt \right]^2 \right)^{1/2}.$$

□

Applying now the proven lemma, we complete the proof of the theorem, since m runs from $-\infty$ to ∞ and we can write 2^m instead of 2^{-m} . □

4. APPLICATIONS

As an application, we obtain integrability results for (2) and (3) given above, for (3) even in a stronger, asymptotic form.

We can relate this problem to a similar one for Fourier transforms as follows. First, given series (2) or (3) with the null sequence of coefficients being in an appropriate sequence space, set for $x \in [n, n+1]$

$$\begin{aligned} A(x) &= a_n + (n-x)\Delta a_n, & a_0 &= 0, \\ B(x) &= b_n + (n-x)\Delta b_n. \end{aligned}$$

So, we construct a corresponding function by means of linear interpolation of the sequence of coefficients. Of course, one may interpolate not only linearly, but there are no problems where this might be of importance so far.

Secondly, for functions of bounded variation φ , to pass from series to integrals and vice versa, we will make use of the following result due to Trigub [14, Th. 4] (see also [15]; an earlier version, for functions with compact support, is due to Belinsky [3]):

$$(21) \quad \sup_{0 < |x| \leq \pi} \left| \int_{-\infty}^{+\infty} \varphi(t) e^{-ixt} dt - \sum_{k=-\infty}^{+\infty} \varphi(k) e^{-ikx} \right| \lesssim \|\varphi\|_{BV}.$$

The relation (21) allows us to pass from estimating trigonometric series (2) and (3) to estimating the Fourier transform of $A(t)$ and $B(t)$, respectively.

Theorem 4. *If the coefficients $\{a_n\}$ in (2) and $\{b_n\}$ in (3) tend to 0 as $n \rightarrow \infty$, and the sequences $\{\Delta a_n\}$ and $\{\Delta b_n\}$ are in $a_{1,2}$, then (2) represents an integrable function on $[0, \pi]$, and*

$$(22) \quad \sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{x} B\left(\frac{\pi}{2x}\right) + \Gamma(x),$$

where $\int_0^{\pi} |\Gamma(x)| dx \lesssim \|\{\Delta b_n\}\|_{a_{1,2}}$.

Proof. First of all, we observe that, as usual, no finite number of coefficients can affect the behavior of trigonometric series. Therefore, we have to estimate

$$(23) \quad \sum_{m=1}^{\infty} \left\{ \sum_{j=1}^{\infty} \left[\sum_{n=j2^{-m}}^{(j+1)2^{-m}-1} \int_n^{n+1} |g(t)| dt \right]^2 \right\}^{1/2} < \infty.$$

By (21) we can deal with the functions $A(x)$ and $B(x)$ instead of the given sequences. Taking in (23) $g(t) = A'(t)$ that equals $-\Delta a_n$ when $n \leq t < n+1$, we prove the first part of the theorem by fulfilling routine calculations.

We treat the remainder term in (22) in the same manner. Finally, to make sure that (6) follow from the obtained asymptotic representation, we get

$$\int_1^N \frac{1}{x} |B\left(\frac{\pi}{2x}\right)| dx = \sum_{n=1}^{N-1} \int_n^{n+1} \frac{1}{x} |B\left(\frac{\pi}{2x}\right)| dx = \sum_{n=1}^{N-1} \int_{\frac{\pi}{2(n+1)}}^{\frac{\pi}{2n}} \frac{1}{t} |B(t)| dt.$$

The right-hand side is obviously equivalent to $\sum_{n=1}^N \frac{|b_n|}{n}$. Letting $N \rightarrow \infty$ leads to (6) and completes the proof. \square

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